

# CONTINUOUS SPECTRUM FOR A CLASS OF NONHOMOGENEOUS DIFFERENTIAL OPERATORS <sup>\*</sup>

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**ABSTRACT.** We study the boundary value problem  $-\operatorname{div}(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}\nabla u) = \lambda|u|^{q(x)-2}u$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with smooth boundary,  $\lambda$  is a positive real number, and the continuous functions  $p_1$ ,  $p_2$ , and  $q$  satisfy  $1 < p_2(x) < q(x) < p_1(x) < N$  and  $\max_{y \in \overline{\Omega}} q(y) < \frac{Np_2(x)}{N-p_2(x)}$  for any  $x \in \overline{\Omega}$ . The main result of this paper establishes the existence of two positive constants  $\lambda_0$  and  $\lambda_1$  with  $\lambda_0 \leq \lambda_1$  such that any  $\lambda \in [\lambda_1, \infty)$  is an eigenvalue, while any  $\lambda \in (0, \lambda_0)$  is not an eigenvalue of the above problem.

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## 1 Introduction and preliminary results

In this paper we are concerned with the study of the eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}\nabla u) = \lambda|u|^{q(x)-2}u, & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) is a bounded domain with smooth boundary,  $\lambda > 0$  is a real number, and  $p_1$ ,  $p_2$ ,  $q$  are continuous functions on  $\overline{\Omega}$ .

The study of eigenvalue problems involving operators with variable exponents growth conditions has captured a special attention in the last few years. This is in keeping with the fact that operators which arise in such kind of problems, like the  $p(x)$ -Laplace operator (i.e.,  $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ , where  $p(x)$  is a continuous positive function), are not homogeneous and thus, a large number of techniques which

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can be applied in the homogeneous case (when  $p(x)$  is a positive constant) fail in this new setting. A typical example is the Lagrange multiplier theorem, which does not apply to the eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{q(x)-2}u, & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \quad (2)$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain. This is due to the fact that the associated Rayleigh quotient is not homogeneous, provided both  $p$  and  $q$  are not constant.

On the other hand, problems like (2) have been largely considered in the literature in the recent years. We give in what follows a concise but complete image of the actual stage of research on this topic.

- In the case when  $p(x) = q(x)$  on  $\overline{\Omega}$ , Fan, Zhang and Zhao [8] established the existence of infinitely many eigenvalues for problem (2) by using an argument based on the Ljusternik-Schnirelmann critical point theory. Denoting by  $\Lambda$  the set of all nonnegative eigenvalues, Fan, Zhang and Zhao showed that  $\Lambda$  is discrete,  $\sup \Lambda = +\infty$  and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function  $p(x)$ , we have  $\inf \Lambda > 0$  (this is in contrast with the case when  $p(x)$  is a constant; then, we always have  $\inf \Lambda > 0$ ).

- In the case when  $\min_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x)$  and  $q(x)$  has a subcritical growth Mihăilescu and Rădulescu [12] used the Ekeland's variational principle in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin.

- In the case when  $\max_{x \in \overline{\Omega}} p(x) < \min_{x \in \overline{\Omega}} q(x)$  and  $q(x)$  has a subcritical growth a mountain-pass argument, similar with those used by Fan and Zhang in the proof of Theorem 4.7 in [7], can be applied in order to show that any  $\lambda > 0$  is an eigenvalue of problem (2).

- In the case when  $\max_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x)$  it can be proved that the energy functional associated to problem (2) has a nontrivial minimum for any positive  $\lambda$  large enough (see Theorem 4.7 in [7]). Clearly, in this case the result in [12] can be also applied. Consequently, in this situation there exist two positive constants  $\lambda^*$  and  $\lambda^{**}$  such that any  $\lambda \in (0, \lambda^*) \cup (\lambda^{**}, \infty)$  is an eigenvalue of problem (2).

In this paper we study problem (1) under the following assumptions:

$$1 < p_2(x) < \min_{y \in \overline{\Omega}} q(y) \leq \max_{y \in \overline{\Omega}} q(y) < p_1(x) < N, \quad \forall x \in \overline{\Omega} \quad (3)$$

and

$$\max_{y \in \overline{\Omega}} q(y) < \frac{Np_2(x)}{N - p_2(x)}, \quad \forall x \in \overline{\Omega}. \quad (4)$$

Thus, the case considered here is different from all the cases studied before. In this new situation we will show the existence of two positive constants  $\lambda_0$  and  $\lambda_1$  with  $\lambda_0 \leq \lambda_1$  such that any  $\lambda \in [\lambda_1, \infty)$  is an eigenvalue of problem (1) while any  $\lambda \in (0, \lambda_0)$  is not an eigenvalue of problem (1). An important

consequence of our study is that, under hypotheses (3) and (4), we have

$$\inf_{u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx} > 0.$$

That fact is proved by using the Lagrange Multiplier Theorem. The absence of homogeneity will be balanced by the fact that assumptions (3) and (4) yield

$$\lim_{\|u\|_{p_1(x)} \rightarrow 0} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx} = \infty$$

and

$$\lim_{\|u\|_{p_1(x)} \rightarrow \infty} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx} = \infty,$$

where  $\|\cdot\|_{p_1(x)}$  stands for the norm in the variable exponent Sobolev space  $W_0^{1,p_1(x)}(\Omega)$ .

We start with some preliminary basic results on the theory of Lebesgue–Sobolev spaces with variable exponent. For more details we refer to the book by Musielak [14] and the papers by Edmunds et al. [4, 5, 6], Kovacik and Rákosník [10], Mihăilescu and Rădulescu [11, 13], and Samko and Vakulov [16].

Set

$$C_+(\overline{\Omega}) = \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any  $h \in C_+(\overline{\Omega})$  we define

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

For any  $p \in C_+(\overline{\Omega})$ , we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u; u \text{ is a measurable real-valued function such that } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

We define on this space the *Luxemburg norm* by

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Let  $L^{p'(x)}(\Omega)$  denote the conjugate space of  $L^{p(x)}(\Omega)$ , where  $1/p(x) + 1/p'(x) = 1$ . For any  $u \in L^{p(x)}(\Omega)$  and  $v \in L^{p'(x)}(\Omega)$  the Hölder type inequality

$$\left| \int_{\Omega} uv dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \quad (5)$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the  $L^{p(x)}(\Omega)$  space, which is the mapping  $\rho_{p(x)} : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

If  $(u_n)$ ,  $u \in L^{p(x)}(\Omega)$  then the following relations hold true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} \quad (6)$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} \quad (7)$$

$$|u_n - u|_{p(x)} \rightarrow 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \rightarrow 0. \quad (8)$$

Next, we define  $W_0^{1,p(x)}(\Omega)$  as the closure of  $C_0^\infty(\Omega)$  under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The space  $W_0^{1,p(x)}(\Omega)$  is a separable and reflexive Banach space. We note that if  $s \in C_+(\overline{\Omega})$  and  $s(x) < p^*(x)$  for all  $x \in \overline{\Omega}$  then the embedding  $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$  is compact and continuous, where  $p^*(x) = \frac{Np(x)}{N-p(x)}$  if  $p(x) < N$  or  $p^*(x) = +\infty$  if  $p(x) \geq N$ .

For applications of Sobolev spaces with variable exponent we refer to Acerbi and Mingione [1], Chen, Levine and Rao [2], Diening [3], Halsey [9], Ruzicka [15], and Zhikov [18]).

## 2 The main result

Since  $p_2(x) < p_1(x)$  for any  $x \in \overline{\Omega}$  it follows that  $W_0^{1,p_1(x)}(\Omega)$  is continuously embedded in  $W_0^{1,p_2(x)}(\Omega)$ . Thus, a solution for a problem of type (1) will be sought in the variable exponent space  $W_0^{1,p_1(x)}(\Omega)$ .

We say that  $\lambda \in \mathbb{R}$  is an *eigenvalue* of problem (1) if there exists  $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$  such that

$$\int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v \, dx - \lambda \int_{\Omega} |u|^{q(x)-2} uv \, dx = 0,$$

for all  $v \in W_0^{1,p_1(x)}(\Omega)$ . We point out that if  $\lambda$  is an eigenvalue of problem (1) then the corresponding eigenfunction  $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$  is a *weak solution* of problem (1).

Define

$$\lambda_1 := \inf_{u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} \, dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} \, dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx}.$$

Our main result is given by the following theorem.

**Theorem 1.** *Assume that conditions (3) and (4) are fulfilled. Then  $\lambda_1 > 0$ . Moreover, any  $\lambda \in [\lambda_1, \infty)$  is an eigenvalue of problem (1). Furthermore, there exists a positive constant  $\lambda_0$  such that  $\lambda_0 \leq \lambda_1$  and any  $\lambda \in (0, \lambda_0)$  is not an eigenvalue of problem (1).*

*Proof.* Let  $E$  denote the generalized Sobolev space  $W_0^{1,p_1(x)}(\Omega)$ . We denote by  $\|\cdot\|$  the norm on  $W_0^{1,p_1(x)}(\Omega)$  and by  $\|\cdot\|_1$  the norm on  $W_0^{1,p_2(x)}(\Omega)$ .

Define the functionals  $J, I, J_1, I_1 : E \rightarrow \mathbb{R}$  by

$$\begin{aligned} J(u) &= \int_{\Omega} \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_{\Omega} \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx, \\ I(u) &= \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx, \\ J_1(u) &= \int_{\Omega} |\nabla u|^{p_1(x)} dx + \int_{\Omega} |\nabla u|^{p_2(x)} dx, \\ I_1(u) &= \int_{\Omega} |u|^{q(x)} dx. \end{aligned}$$

Standard arguments imply that  $J, I \in C^1(E, \mathbb{R})$  and for all  $u, v \in E$ ,

$$\begin{aligned} \langle J'(u), v \rangle &= \int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v dx, \\ \langle I'(u), v \rangle &= \int_{\Omega} |u|^{q(x)-2} uv dx. \end{aligned}$$

We split the proof of Theorem 1 into four steps.

- STEP 1. We show that  $\lambda_1 > 0$ .

Since for any  $x \in \overline{\Omega}$  we have  $p_1(x) > q^+ \geq q(x) \geq q^- > p_2(x)$  we deduce that for any  $u \in E$ ,

$$2(|\nabla u(x)|^{p_1(x)} + |\nabla u(x)|^{p_2(x)}) \geq |\nabla u(x)|^{q^+} + |\nabla u(x)|^{q^-}$$

and

$$|u(x)|^{q^+} + |u(x)|^{q^-} \geq |u(x)|^{q(x)}.$$

Integrating the above inequalities we find

$$2 \int_{\Omega} (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}) dx \geq \int_{\Omega} (|\nabla u|^{q^+} + |\nabla u|^{q^-}) dx, \quad \forall u \in E \quad (9)$$

and

$$\int_{\Omega} (|u|^{q^+} + |u|^{q^-}) dx \geq \int_{\Omega} |u|^{q(x)} dx, \quad \forall u \in E. \quad (10)$$

By Sobolev embeddings, there exist positive constants  $\lambda_{q^+}$  and  $\lambda_{q^-}$  such that

$$\int_{\Omega} |\nabla u|^{q^+} dx \geq \lambda_{q^+} \int_{\Omega} |u|^{q^+} dx, \quad \forall u \in W_0^{1,q^+}(\Omega) \quad (11)$$

and

$$\int_{\Omega} |\nabla u|^{q^-} dx \geq \lambda_{q^-} \int_{\Omega} |u|^{q^-} dx, \quad \forall u \in W_0^{1,q^-}(\Omega). \quad (12)$$

Using again the fact that  $q^- \leq q^+ < p_1(x)$  for any  $x \in \overline{\Omega}$  we deduce that  $E$  is continuously embedded in  $W_0^{1,q^+}(\Omega)$  and in  $W_0^{1,q^-}(\Omega)$ . Thus, inequalities (11) and (12) hold true for any  $u \in E$ .

Using inequalities (11), (12) and (10) it is clear that there exists a positive constant  $\mu$  such that

$$\int_{\Omega} (|\nabla u|^{q^+} + |\nabla u|^{q^-}) dx \geq \mu \int_{\Omega} |u|^{q(x)} dx, \quad \forall u \in E. \quad (13)$$

Next, inequalities (13) and (9) yield

$$\int_{\Omega} (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}) dx \geq \frac{\mu}{2} \int_{\Omega} |u|^{q(x)} dx, \quad \forall u \in E. \quad (14)$$

By relation (14) we deduce that

$$\lambda_0 = \inf_{v \in E \setminus \{0\}} \frac{J_1(v)}{I_1(v)} > 0 \quad (15)$$

and thus,

$$J_1(u) \geq \lambda_0 I_1(u), \quad \forall u \in E. \quad (16)$$

The above inequality yields

$$p_1^+ \cdot J(u) \geq J_1(u) \geq \lambda_0 I_1(u) \geq \lambda_0 I(u) \quad \forall u \in E. \quad (17)$$

The last inequality assures that  $\lambda_1 > 0$  and thus, step 1 is verified.

• **STEP 2.** We show that  $\lambda_1$  is an eigenvalue of problem (1).

**Lemma 1.** *The following relations hold true:*

$$\lim_{\|u\| \rightarrow \infty} \frac{J(u)}{I(u)} = \infty \quad (18)$$

and

$$\lim_{\|u\| \rightarrow 0} \frac{J(u)}{I(u)} = \infty. \quad (19)$$

*Proof.* Since  $E$  is continuously embedded in  $L^{q^\pm}(\Omega)$  it follows that there exist two positive constants  $c_1$  and  $c_2$  such that

$$\|u\| \geq c_1 \cdot |u|_{q^+}, \quad \forall u \in E \quad (20)$$

and

$$\|u\| \geq c_2 \cdot |u|_{q^-}, \quad \forall u \in E. \quad (21)$$

For any  $u \in E$  with  $\|u\| > 1$  by relations (6), (10), (20), (21) we infer

$$\frac{J(u)}{I(u)} \geq \frac{\frac{\|u\|^{p_1^-}}{p_1^+}}{\frac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\frac{\|u\|^{p_1^-}}{p_1^+}}{\frac{c_1^{-q^+} \|u\|^{q^+} + c_2^{-q^-} \|u\|^{q^-}}{q^-}}.$$

Since  $p_1^- > q^+ \geq q^-$ , passing to the limit as  $\|u\| \rightarrow \infty$  in the above inequality we deduce that relation (18) holds true.

Next, let us remark that since  $p_1(x) > p_2(x)$  for any  $x \in \overline{\Omega}$ , the space  $W_0^{1,p_1(x)}(\Omega)$  is continuously embedded in  $W_0^{1,p_2(x)}(\Omega)$ . Thus, if  $\|u\| \rightarrow 0$  then  $\|u\|_1 \rightarrow 0$ .

The above remarks enable us to affirm that for any  $u \in E$  with  $\|u\| < 1$  small enough we have  $\|u\|_1 < 1$ .

On the other hand, since (4) holds true we deduce that  $W_0^{1,p_2(x)}(\Omega)$  is continuously embedded in  $L^{q^\pm}(\Omega)$ . It follows that there exist two positive constants  $d_1$  and  $d_2$  such that

$$\|u\|_1 \geq d_1 \cdot |u|_{q^+}, \quad \forall u \in W_0^{1,p_2(x)}(\Omega) \quad (22)$$

and

$$\|u\|_1 \geq d_2 \cdot |u|_{q^-}, \quad \forall u \in W_0^{1,p_2(x)}(\Omega). \quad (23)$$

Thus, for any  $u \in E$  with  $\|u\| < 1$  small enough, relations (7), (10), (22), (23) imply

$$\frac{J(u)}{I(u)} \geq \frac{\frac{\int_{\Omega} |\nabla u|^{p_2(x)} dx}{p_2^+}}{\frac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\frac{\|u\|_1^{p_2^+}}{p_2^+}}{\frac{d_1^{-q^+} \|u\|_1^{q^+} + d_2^{-q^-} \|u\|_1^{q^-}}{q^-}}.$$

Since  $p_2^+ < q^- \leq q^+$ , passing to the limit as  $\|u\| \rightarrow 0$  (and thus,  $\|u\|_1 \rightarrow 0$ ) in the above inequality we deduce that relation (19) holds true. The proof of Lemma 1 is complete.  $\square$

**Lemma 2.** *There exists  $u \in E \setminus \{0\}$  such that  $\frac{J(u)}{I(u)} = \lambda_1$ .*

*Proof.* Let  $\{u_n\} \subset E \setminus \{0\}$  be a minimizing sequence for  $\lambda_1$ , that is,

$$\lim_{n \rightarrow \infty} \frac{J(u_n)}{I(u_n)} = \lambda_1 > 0. \quad (24)$$

By relation (18) it is clear that  $\{u_n\}$  is bounded in  $E$ . Since  $E$  is reflexive it follows that there exists  $u \in E$  such that  $u_n$  converges weakly to  $u$  in  $E$ . On the other hand, similar arguments as those used in the proof of Lemma 3.4 in [11] show that the functional  $J$  is weakly lower semi-continuous. Thus, we find

$$\liminf_{n \rightarrow \infty} J(u_n) \geq J(u). \quad (25)$$

By relation (4) it follows that  $E$  is compactly embedded in  $L^{q(x)}(\Omega)$ . Thus,  $u_n$  converges strongly in  $L^{q(x)}(\Omega)$ . Then, by relation (8) it follows that

$$\lim_{n \rightarrow \infty} I(u_n) = I(u). \quad (26)$$

Relations (25) and (26) imply that if  $u \neq 0$  then

$$\frac{J(u)}{I(u)} = \lambda_1.$$

Thus, in order to conclude that the lemma holds true it is enough to show that  $u$  is not trivial. Assume by contradiction the contrary. Then  $u_n$  converges weakly to 0 in  $E$  and strongly in  $L^{q(x)}(\Omega)$ . In other words, we will have

$$\lim_{n \rightarrow \infty} I(u_n) = 0. \quad (27)$$

Letting  $\epsilon \in (0, \lambda_1)$  be fixed by relation (24) we deduce that for  $n$  large enough we have

$$|J(u_n) - \lambda_1 I(u_n)| < \epsilon I(u_n),$$

or

$$(\lambda_1 - \epsilon)I(u_n) < J(u_n) < (\lambda_1 + \epsilon)I(u_n).$$

Passing to the limit in the above inequalities and taking into account that relation (27) holds true we find

$$\lim_{n \rightarrow \infty} J(u_n) = 0.$$

That fact combined with relation (8) implies that actually  $u_n$  converges strongly to 0 in  $E$ , i.e.  $\lim_{n \rightarrow \infty} \|u_n\| = 0$ . By this information and relation (19) we get

$$\lim_{n \rightarrow \infty} \frac{J(u_n)}{I(u_n)} = \infty,$$

and this is a contradiction. Thus,  $u \neq 0$ . The proof of Lemma 2 is complete.  $\square$

By Lemma 2 we conclude that there exists  $u \in E \setminus \{0\}$  such that

$$\frac{J(u)}{I(u)} = \lambda_1 = \inf_{w \in E \setminus \{0\}} \frac{J(w)}{I(w)}. \quad (28)$$

Then, for any  $v \in E$  we have

$$\frac{d}{d\epsilon} \frac{J(u + \epsilon v)}{I(u + \epsilon v)} \Big|_{\epsilon=0} = 0.$$

A simple computation yields

$$\int_{\Omega} (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v \, dx \cdot I(u) - J(u) \cdot \int_{\Omega} |u|^{q(x)-2} u v \, dx = 0, \quad \forall v \in E. \quad (29)$$

Relation (29) combined with the fact that  $J(u) = \lambda_1 I(u)$  and  $I(u) \neq 0$  implies the fact that  $\lambda_1$  is an eigenvalue of problem (1). Thus, step 2 is verified.

• **STEP 3.** We show that any  $\lambda \in (\lambda_1, \infty)$  is an eigenvalue of problem (1).

Let  $\lambda \in (\lambda_1, \infty)$  be arbitrary but fixed. Define  $T_\lambda : E \rightarrow \mathbb{R}$  by

$$T_\lambda(u) = J(u) - \lambda I(u).$$

Clearly,  $T_\lambda \in C^1(E, \mathbb{R})$  with

$$\langle T'_\lambda(u), v \rangle = \langle J'(u), v \rangle - \lambda \langle I'(u), v \rangle, \quad \forall u \in E.$$



Thus,  $\lambda$  is an eigenvalue of problem (1) if and only if there exists  $u_\lambda \in E \setminus \{0\}$  a critical point of  $T_\lambda$ .

With similar arguments as in the proof of relation (18) we can show that  $T_\lambda$  is coercive, i.e.  $\lim_{\|u\| \rightarrow \infty} T_\lambda(u) = \infty$ . On the other hand, as we have already remarked, similar arguments as those used in the proof of Lemma 3.4 in [11] show that the functional  $T_\lambda$  is weakly lower semi-continuous. These two facts enable us to apply Theorem 1.2 in [17] in order to prove that there exists  $u_\lambda \in E$  a global minimum point of  $T_\lambda$  and thus, a critical point of  $T_\lambda$ . In order to conclude that step 4 holds true it is enough to show that  $u_\lambda$  is not trivial. Indeed, since  $\lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{J(u)}{I(u)}$  and  $\lambda > \lambda_1$  it follows that there exists  $v_\lambda \in E$  such that

$$J(v_\lambda) < \lambda I(v_\lambda),$$

or

$$T_\lambda(v_\lambda) < 0.$$

Thus,

$$\inf_E T_\lambda < 0$$

and we conclude that  $u_\lambda$  is a nontrivial critical point of  $T_\lambda$ , or  $\lambda$  is an eigenvalue of problem (1). Thus, step 3 is verified.

• **STEP 4.** Any  $\lambda \in (0, \lambda_0)$ , where  $\lambda_0$  is given by (15), is not an eigenvalue of problem (1).

Indeed, assuming by contradiction that there exists  $\lambda \in (0, \lambda_0)$  an eigenvalue of problem (1) it follows that there exists  $u_\lambda \in E \setminus \{0\}$  such that

$$\langle J'(u_\lambda), v \rangle = \lambda \langle I'(u_\lambda), v \rangle, \quad \forall v \in E.$$

Thus, for  $v = u_\lambda$  we find

$$\langle J'(u_\lambda), u_\lambda \rangle = \lambda \langle I'(u_\lambda), u_\lambda \rangle,$$

that is,

$$J_1(u_\lambda) = \lambda I_1(u_\lambda).$$

The fact that  $u_\lambda \in E \setminus \{0\}$  assures that  $I_1(u_\lambda) > 0$ . Since  $\lambda < \lambda_0$ , the above information yields

$$J_1(u_\lambda) \geq \lambda_0 I_1(u_\lambda) > \lambda I_1(u_\lambda) = J_1(u_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Thus, step 4 is verified.

By steps 2, 3 and 4 we deduce that  $\lambda_0 \leq \lambda_1$ . The proof of Theorem 1 is now complete.  $\square$

**Remark 1.** At this stage we are not able to deduce whether  $\lambda_0 = \lambda_1$  or  $\lambda_0 < \lambda_1$ . In the latter case an interesting question concerns the existence of eigenvalues of problem (1) in the interval  $[\lambda_0, \lambda_1]$ . We propose to the reader the study of these open problems.

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